The Mad Hatters

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- They choose a hat on there own head simultaneously. Win if and only if both choose a white hat
- What is the optimal strategy?

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- Players who guess wrong are eaten, those who guess right get shown the way off the isand.

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□	■	□	□	■
Back				Front

Back to Levine problem

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- Play is simulataneous.
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- This does at least identify what's hard here: We can't share information!

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- Win half the time!!!

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- First less likely hat colour is best.

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- 256 group strategies.

Two hats.

Two hats	■■	■□	□■	□□
■■				
■□				
□■				
□□				

Two hats.

Two hats	■■	■□	□■	□□
■■	lose	lose	lose	lose
■□	lose			
□■	lose			
□□	lose			win

Two hats.

Two hats	■■	■□	□■	□□
■■	lose	lose	lose	lose
■□	lose			share
□■	lose			share
□□	lose	share	share	win

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Two hats	■■	■□	□■	□□
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■□	lose	win	lose	lose
□■	lose	lose	win	win
□□	lose	lose	win	win

Three hats.

Three hats	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
Picture	■■■	□■■	■□■	□□■	■■□	□■□	■□□	□□□
Choice	any	1	3	1	2	2	3	any

Table : Optimal strategy on 3 hats

For convience

White hats	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
Picture	■■■	□■■	■□■	□□■	■■□	□■□	■□□	□□□
Choice	1	1	3	1	2	2	3	1

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Three hats-example.

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Table : Optimal strategy on 3 hats

□	■
■	□
□	□
Player 1	Player 2

Table : Player 1 chooses hat 1, Player 2 hat 2. They win

Three hats.

White hats	■■■	■■□	■□■	■□□	□■■	□■□	□□■	□□□
■■■								
■■□			w	w				
■□■		w				w		
■□□		w		w		w		
□■■					w		w	w
□■□		w	w				w	w
□□■					w	w	w	w
□□□					w	w	w	w

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- Wait a lot of generations

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- Took there "natural" infinite analogs.

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- S_3 Dual of S_1. Toggle all colours and play S_1

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Computing Performance

$$\begin{aligned}
V_{S^d}(p) &= \mathbb{P}(A_{S^d}^{W,W}(p)) \\
&= \mathbb{P}(A_S^{B,B}(q)) \\
&= p - \mathbb{P}(A_S^{B,W}(q)) \\
&= p - (q - \mathbb{P}(A_S^{W,W}(q))) \\
&= p - q + \mathbb{P}(A_S^{W,W}(q)) \\
&= 2p - 1 + V_S(q)
\end{aligned}$$

Performance

For our game with probability p of each hat being white, this strategy gives the following lower bound on $V(p)$:

1. $\dfrac{p(1+p+p^2+3p^3-3p^4+p^5)}{(1+p)(2-p)(1+p^2)} \leq V(p)$ for $p \leq \frac{1}{2}$;
2. $\dfrac{p(1+5p-10p^2+10p^3-5p^4+p^5)}{(2-2p+p^2)(1+p)(2-p)}$ for $\frac{1}{2} \leq p$.

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- Lowest terms of a and b is strongest, works best for $\binom{b}{a}$ small.

Future Work

- Multiple hat colours

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- Open: Does the probability of winning go to zero?
- Both multiple colours and multiple players.

Thanks

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